



Institute
and Faculty
of Actuaries

Compartmental Reserving Models

Jake Morris & Markus Gesmann



Agenda

- Introduction to compartmental reserving modelling framework
- Modelling the mean claims process
- Modelling reserve uncertainty
- Case study using R/Stan & 'brms'
- Summary





Institute
and Faculty
of Actuaries

Compartmental Reserving Models

Introduction to compartmental reserving modelling framework

The compartmental reserving modelling framework

- Key idea: Start by fitting model to data, not data to model
- At the centre of the framework is to think about the data generating process
 - Begin by simulating artificial data that shares the expected real data characteristics
- Use “compartments” to reflect different business processes
 - Exposure being underwritten
 - Claims being reported
 - Payments being made

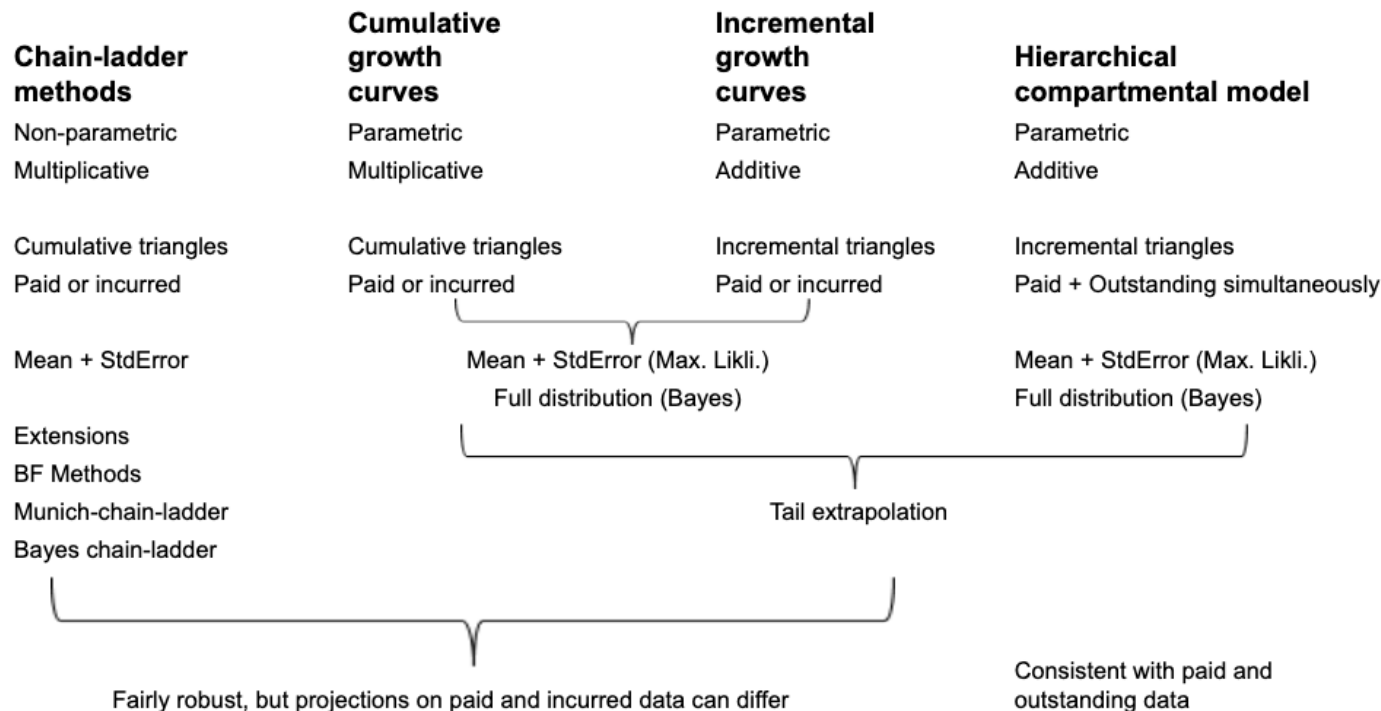


The compartmental reserving modelling framework

- Expert knowledge required to model and parameterise
 - A Framework *not* a Method!
- Benefits:
 - Transparent model that can be criticised
 - Provides additional insight into business processes
 - Practitioner knowledge can be incorporated into model easily



Relation to other models / frameworks



Hierarchical compartmental reserving models in a nutshell

- Use differential equations to model the mean claims process through time
- Consider which data generating distribution gave rise to the mean process, e.g. Gaussian, Log-normal, Negative-binomial, Tweedie
 - Which variance metric can be considered constant across claims development periods, if any? E.g. coefficient of variation
- Use expert knowledge to set priors on parameters
- Generate data from model: do simulations capture expected features?
- Update model with actual observations to obtain posterior parameter estimates and predictive distributions





Institute
and Faculty
of Actuaries

Compartmental Reserving Models

Modelling mean claims process

Compartmental models

- Popular tool in multiple disciplines to describe the behaviour and dynamics of interacting processes using differential equations
- Each compartment relates to a different stage or population of the process, usually modelled with its own differential equation
- Examples are found in:
 - Pharma, to model how drugs interact with the body
 - Electric engineering, to describe the flow of electricity
 - Biophysics, to explain the interactions of neurons
 - Epidemiology, to understand the spread of diseases
 - Biology, to describe interaction of different populations



Simple Compartmental claims development model

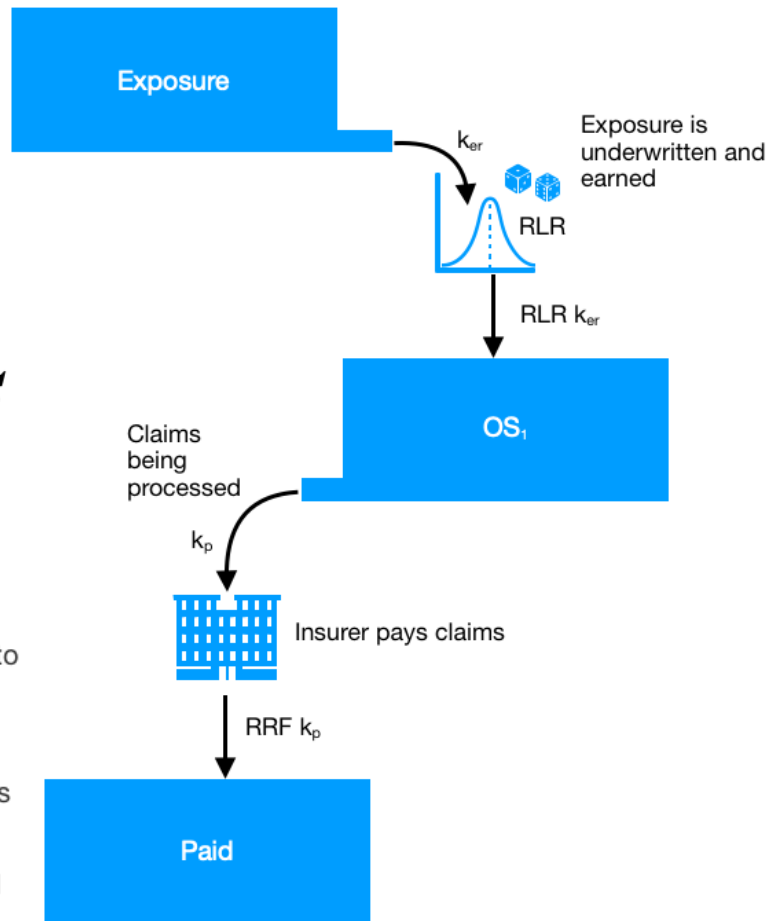
$$dEX/dt = -k_{er} \cdot EX$$

$$dOS/dt = k_{er} \cdot RLR \cdot EX - k_p \cdot OS$$

$$dPD/dt = k_p \cdot RRF \cdot OS$$

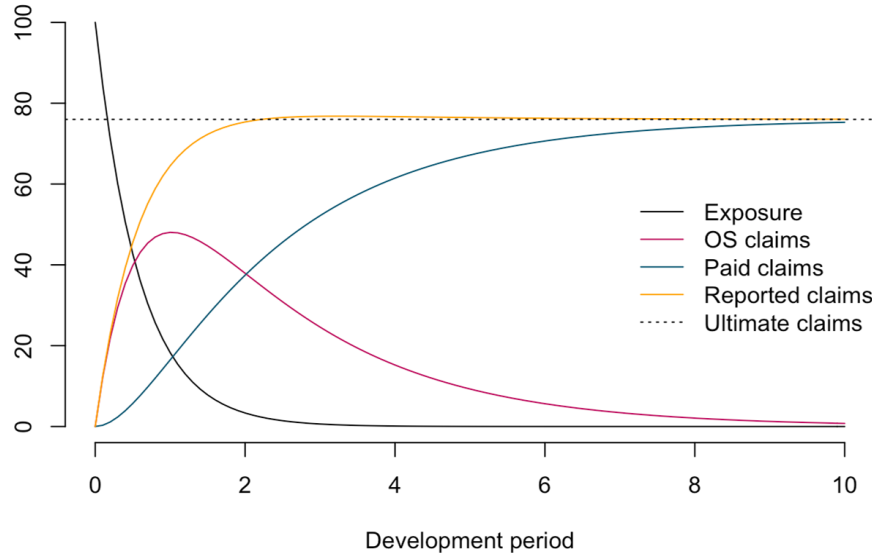
The parameters describe:

- k_{er} the rate at which claim events occur and are subsequently reported to the insurer
- RLR the reported loss ratio
- RRF the reserve robustness factor, the proportion of outstanding claims that eventually are paid
- k_p the rate of payment, i.e. the rate at which outstanding claims are paid



Analytical solutions can be derived by iterative integration

Solutions Define Development Patterns



$$dEX/dt = -k_{er} \cdot EX$$

$$dOS/dt = k_{er} \cdot RLR \cdot EX - k_p \cdot OS$$

$$dPD/dt = k_p \cdot RRF \cdot OS$$

$$EX(t) = \Pi \cdot \exp(-k_{er}t)$$

$$OS(t) = \frac{\Pi \cdot RLR \cdot k_{er}}{k_{er} - k_p} \cdot (\exp(-k_p t) - \exp(-k_{er}t))$$

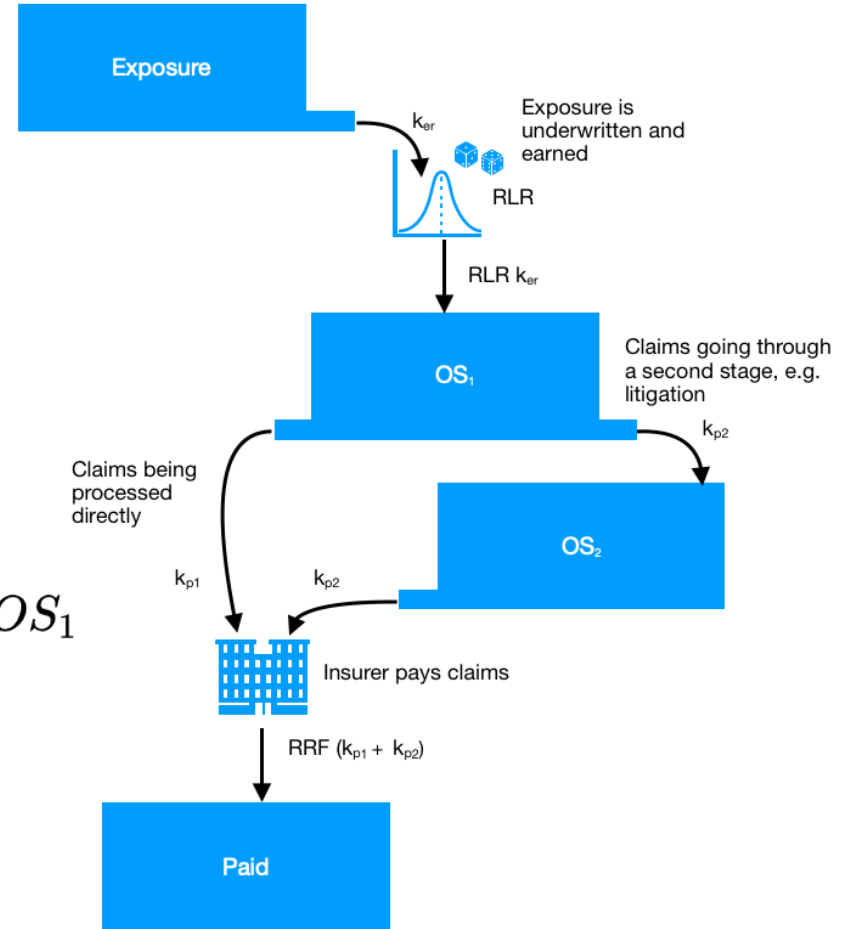
$$PD(t) = \frac{\Pi \cdot RLR \cdot RRF}{k_{er} - k_p} (k_{er} \cdot (1 - \exp(-k_p t)) - k_p \cdot (1 - \exp(-k_{er}t)))$$



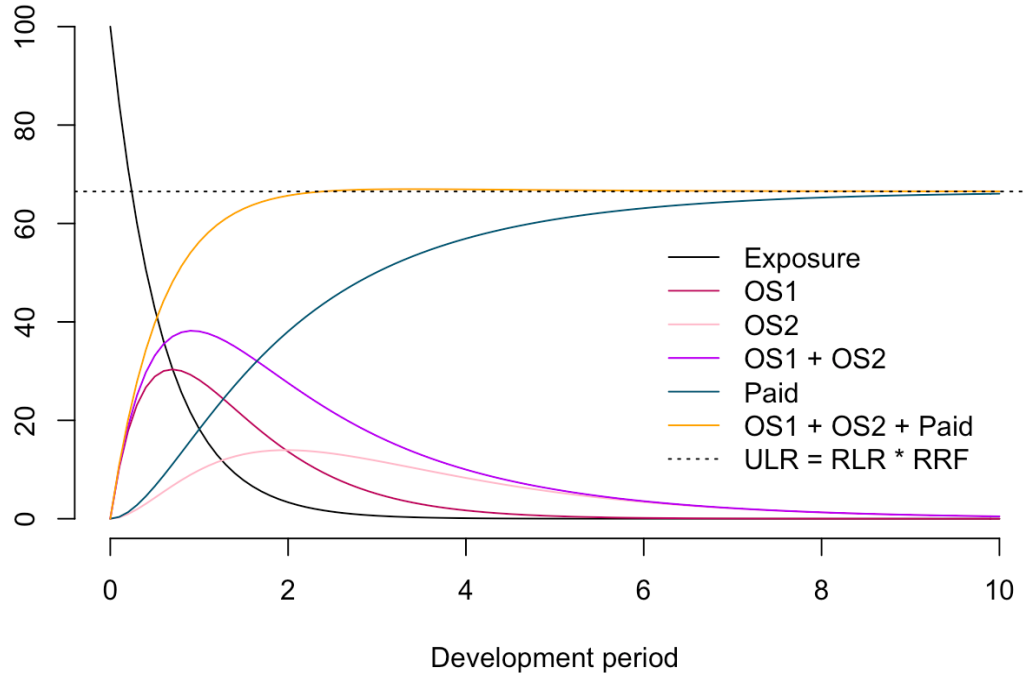
Compartmental model with two claims settlement processes

Single rate of settlement can be too simplistic to capture heterogeneous claims characteristics and hence settlement processes at an aggregated level

$$\begin{aligned}dEX/dt &= -k_{er} \cdot EX \\dOS_1/dt &= k_{er} \cdot RLR \cdot EX - (k_{p_1} + k_{p_2}) \cdot OS_1 \\dOS_2/dt &= k_{p_2} \cdot (OS_1 - OS_2) \\dPD/dt &= RRF \cdot (k_{p_1} \cdot OS_1 + k_{p_2} \cdot OS_2)\end{aligned}$$



Analytical solutions illustrate different processes



$$EX(t) = \Pi \exp(-k_{er}t)$$

$$OS_1(t) = \frac{\Pi RLR k_{er}}{k_{er} - k_{p_1} - k_{p_2}} [\exp(-(k_{p_1} + k_{p_2})t) - \exp(-k_{er}t)]$$

$$OS_2(t) = \frac{\Pi RLR k_{er} k_{p_2}}{k_{p_1}(k_{p_2} - k_{er})(k_{er} - k_{p_1} - k_{p_2})} [\exp(-(k_{p_1} + k_{p_2})t)(k_{er} - k_{p_2}) - \exp(-k_{p_2}t)(k_{er} - k_{p_1} - k_{p_2}) - \exp(-k_{er}t)k_{p_1}]$$

$$PD(t) = \frac{\Pi RLR RRF}{k_{p_1}(k_{p_2} - k_{er})(k_{er} - k_{p_1} - k_{p_2})} [\begin{aligned} &(k_{p_1}(k_{er}(k_{p_1} - k_{er}) - k_{p_2}(k_{p_1} + k_{p_2})) + 2k_{er}k_{p_2}) + \\ &\exp(-(k_{p_1} + k_{p_2})t)(k_{er}(k_{er}k_{p_1} - k_{er}k_{p_2} + k_{p_2}^2 - k_{p_1}k_{p_2})) + \\ &\exp(-k_{p_2}t)(k_{er}k_{p_2}(k_{er} - k_{p_1} - k_{p_2}) + \\ &\exp(-k_{er}t)(k_{p_1}(k_{p_1}k_{p_2} + k_{p_2}^2 - k_{er}k_{p_1})) \end{aligned}]$$



Compartment models can be extended easily ...

- To incorporate different claims processes, e.g. a faster and slower settlement process
- Separate earning and reporting processes
- Time dependent parameters
- Calendar year effects
- Analytical solutions may become complex, but can opt for ODE solvers
- Note: Paid claims are scaled integration of outstanding claims





Institute
and Faculty
of Actuaries

Compartmental Reserving Models

Modelling uncertainty

Be careful with your parameter bookkeeping

- In a Bayesian framework we distinguish between:
 - **Priors**, before we have actual data:
 - Prior parameter distribution, e.g. Planning Loss Ratio (PLR)
 - Prior predictive distribution, e.g. Capital Model Loss Ratio (CLR)
 - **Posteriors**, priors updated with actual data:
 - Posterior parameter distribution, e.g. Expected Loss Ratio (ELR)
 - Posterior predictive distribution, e.g. Ultimate Loss Ratio (ULR)

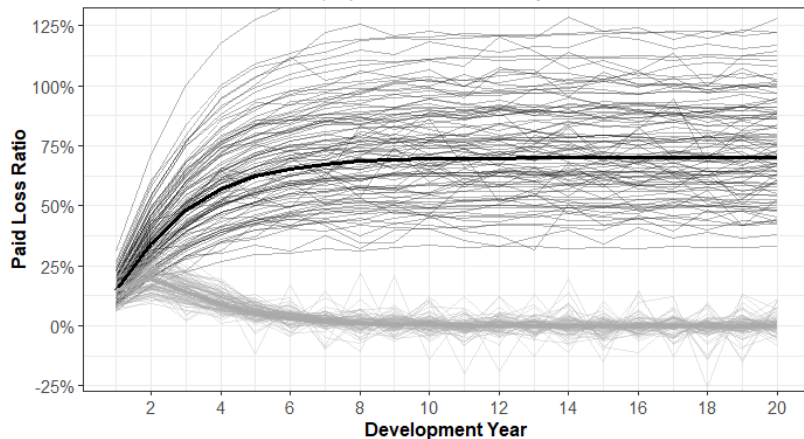


Which process variance metric can be kept constant?

Simulated Behaviour: Cumulative vs. Incremental Model

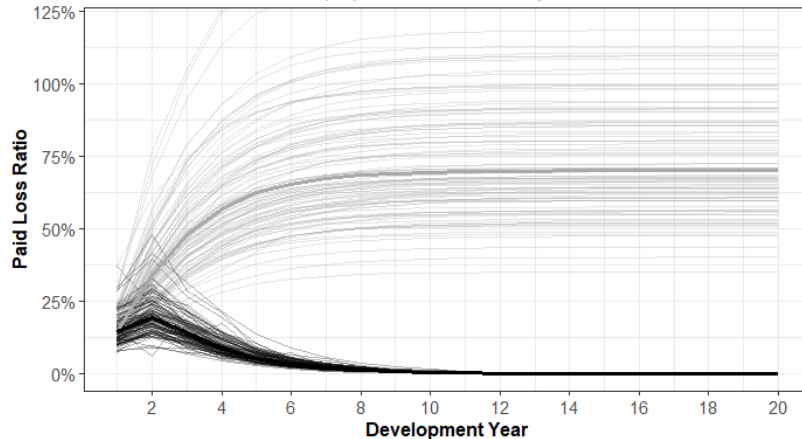
Cumulative Paid Model: Cumulative vs. Incremental Realisations

Modelled Cumulative = Black | Implied Incremental = Grey



Incremental Paid Model: Cumulative vs. Incremental Realisations

Modelled Incremental = Black | Implied Cumulative = Grey

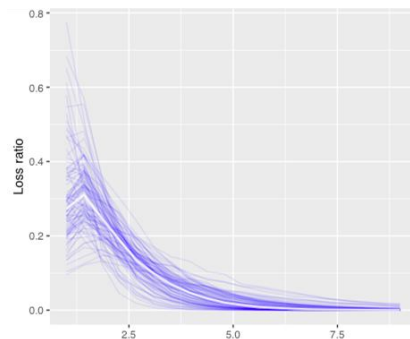
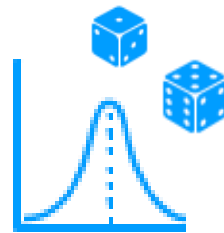


- Modelling cumulative paid data directly is problematic
- Modelling incremental paid with constant CoV more realistic



Parameter Uncertainty + Data Generating Process

- “Which parameters combinations are consistent with our data and model?”
 - Start with prior assumptions, e.g. $ULR \sim \log N(\mu, \sigma)$, ...
 - Update prior assumptions via the likelihood, $L(y; ULR, \dots)$
 - Obtain ‘posterior’ parameter distributions, $p(ULR, \dots | y)$
- From posterior ELR to posterior ULR:
 1. Simulate realisations from posterior parameter distributions
 2. Simulate realisations from assumed data generation distribution
 3. Sum future paid increment posterior predictive paths



‘Borrow Strength’ with Hierarchies

- Which parameters vary across different cohorts, e.g. accident years and which are more likely to be fixed?
 - Chain-ladder assumption: shape of curves considered fixed across accident years
 - Ultimate loss ratios vary by accident years
- Hierarchical models allow all parameters to vary across cohort
 - A parameter has greater potential to deviate from the ‘cohort average’ parameter value where data are rich (credibility weighting / shrinkage)
 - Hierarchical priors are used to prevent overfitting (regularization)





Institute
and Faculty
of Actuaries

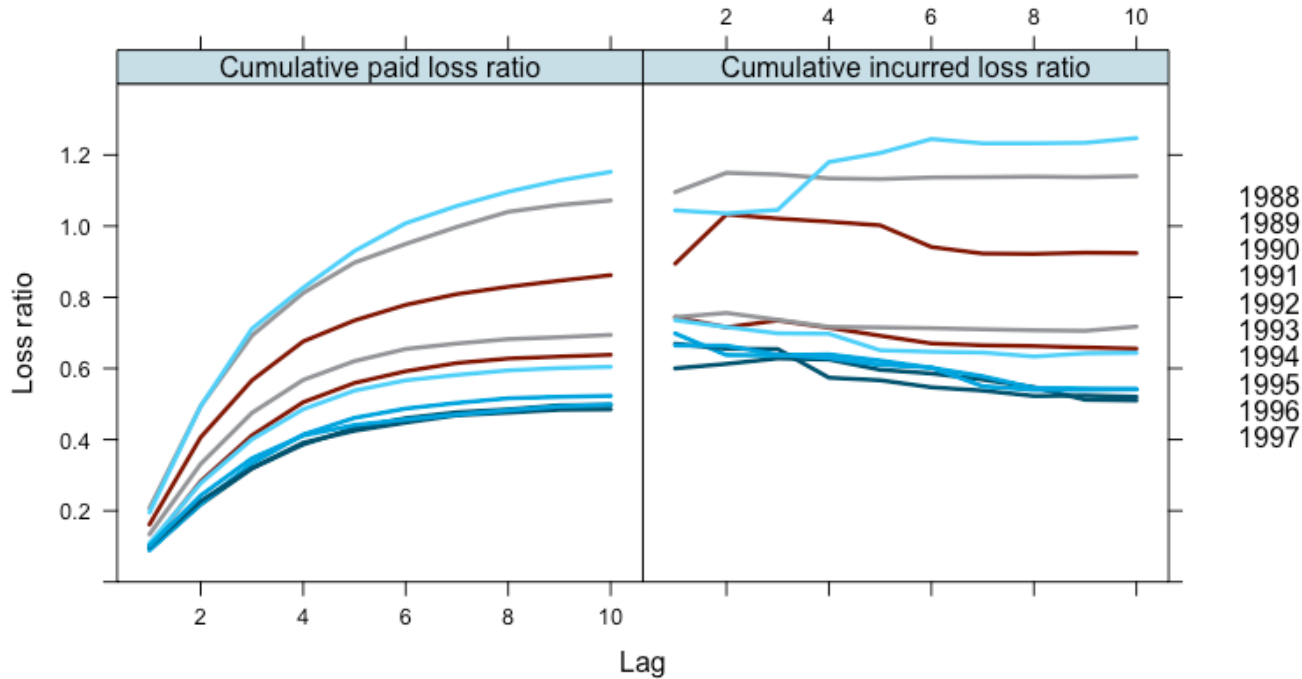
Compartmental Reserving Models

Case Study

25 April 2019

Example data set: Cumulative paid and incurred

California Cas Group



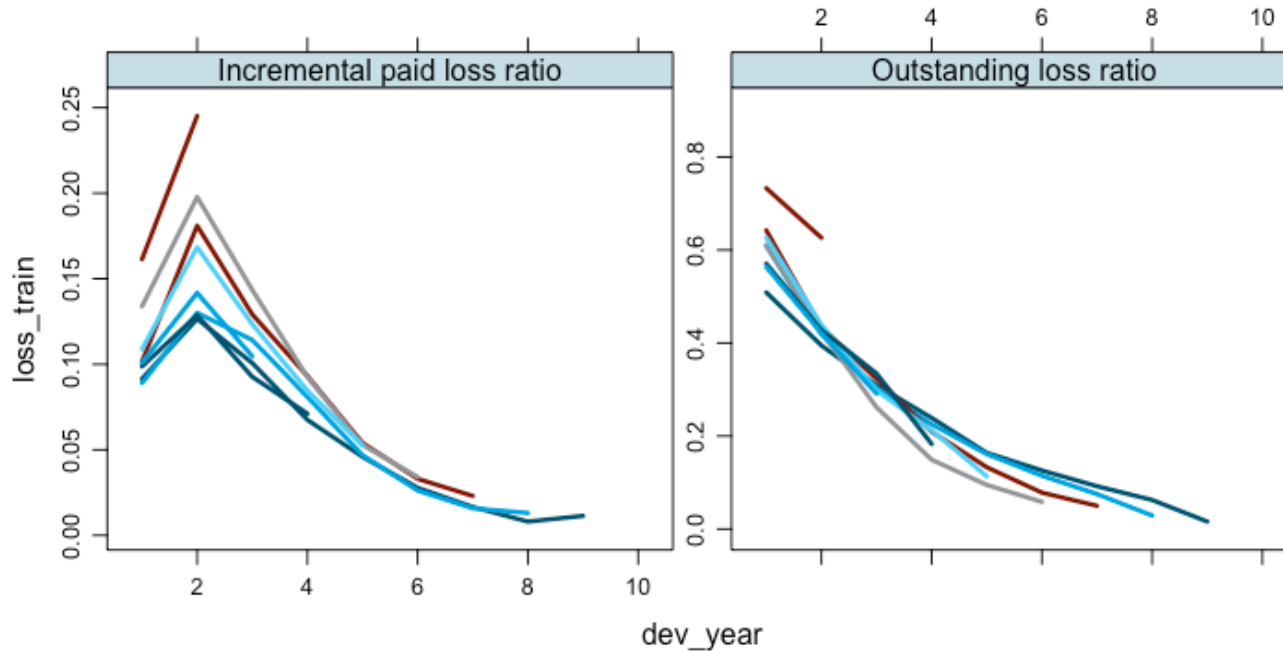
Full 10 year history
for accident years
1988 - 1997



Institute
and Faculty
of Actuaries

Example data set: Incremental paid and outstanding

California Cas Group



Training triangles
for accident years
1988 - 1997

1988
1989
1990
1991
1992
1993
1994
1995
1996
1997



Institute
and Faculty
of Actuaries

Model process and location parameter

- Let t be the development period
- $y(t, \delta)$ describing paid ($\delta = 1$) and outstanding claims ($\delta = 0$)
- Assume process follows a log-normal distribution, with constant CoV_δ
- We model the median of the claims process as:

$$\begin{aligned}y(t_j) &\sim \log \mathcal{N}(\mu(t_j), \sigma_\delta^2) \\ \mu(t_j) &= \log f(t_j; \Theta, \delta) \\ f(t_j; \Theta, \delta) &= (1 - \delta)OS(t_j; \Theta) + \delta (PD(t_j; \Theta) - PD(t_{j-1}; \Theta)) \\ \delta &= \begin{cases} 0 & \text{if } y \text{ is outstanding claim} \\ 1 & \text{if } y \text{ is paid claim} \end{cases}\end{aligned}$$



Setup analytical solution in Stan/C

```
myFuns <- "  
  
real paid(real t, real ker, real kp, real RLR, real RRF){  
  return(RLR*RRF/(ker - kp) * (ker *(1 - exp(-kp*t)) - kp*(1 - exp(-ker*t))));  
}  
  
real os(real t, real ker, real kp, real RLR){  
  return((RLR*ker/(ker - kp) * (exp(-kp*t) - exp(-ker*t))));  
}
```



Setup analytical solution in Stan/C cont'd

```
real claimsprocess(real t, real devfreq, real oker, real okp,  
                  real oRLR, real oRRF, real delta){  
  
  real out;  
  
  real ker = 1 + exp(oker); real kp = 1 * exp(okp * 0.5);  
  real RLR = 0.7 * exp(oRLR * 0.1); real RRF = exp(oRLR * 0.1);  
  
  out = os(t, ker, kp, RLR) * (1 - delta) + paid(t, ker, kp, RLR, RRF) * delta;  
  if( (delta > 0) && (t > devfreq) ){ // paid greater dev period 1  
    out = out - paid(t - devfreq, ker, kp, RLR, RRF)*delta;  
  }  
  
  return(out);  
  
}"
```



Parameter structures

- Parameters assumed 'fixed' across accident years
 - $k_{er}, k_p, \sigma_{pd}, \sigma_{os}$
- Parameters assumed to vary 'randomly' by accident year
 - $RLR_{[i]}, RRF_{[i]}$, allowing for correlation:

$$\begin{pmatrix} RLR_{[i]} \\ RRF_{[i]} \end{pmatrix} \sim \begin{pmatrix} RLR_0 \\ RRF_0 \end{pmatrix} + \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right)$$

with RLR_0, RRF_0 following a log-normal distribution



Create non-linear model formula in R

```
frml <- bf(loss_train ~ log(claimsprocess(dev_year, 1.0, oker, okp,  
                                         oRLR, oRRF, delta)),  
           oRLR ~ 1 + (1 |ID| origin_year),  
           oRRF ~ 1 + (1 |ID| origin_year),  
           oker ~ 1, okp ~ 1,  
           sigma ~ 0 + deltaf,  
           nl = TRUE)
```



Set prior parameter distributions

- Setting sensible priors is crucial for MCMC simulations
- Using standard Gaussians seems to be advisable
 - Standard Gaussian can be transformed to appropriate value ranges
- Example:
 - original oRLR $\sim N(0,1)$
 - Transformed RLR = $0.7 * \exp(\text{oRLR} * 0.1)$, i.e. log-normal distribution



Set prior distribution over parameters

```
myriors <- c(prior(normal(0, 1), nlpar = "oRLR"),  
             prior(normal(0, 1), nlpar = "oRRF"),  
             prior(normal(0, 1), nlpar = "oker"),  
             prior(normal(0, 1), nlpar = "okp"),  
             prior(normal(-3, 0.2), class = "b",  
                    coef="deltafpaid", dpar= "sigma"),  
             prior(normal(-3, 0.2), class = "b",  
                    coef="deltafos", dpar= "sigma"),  
             prior(student_t(10, 0, 0.1), class = "sd", nlpar = "oRLR"),  
             prior(student_t(10, 0, 0.05), class = "sd", nlpar = "oRRF"))
```



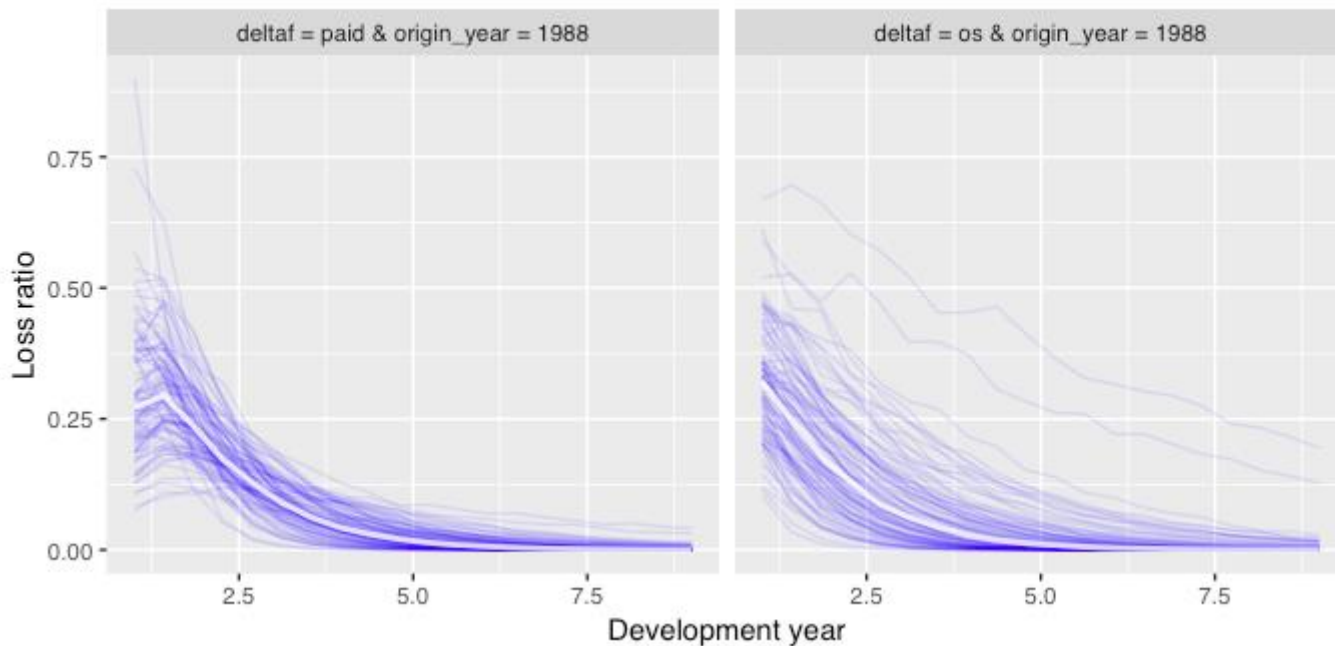
Run prior predictive model with 'brms' in R/Stan

```
bla <- brm(frml, data = myDat,  
  family = brmsfamily("lognormal", link_sigma = "log"),  
  prior = mypriors,  
  control = list(adapt_delta = 0.9, max_treedepth=15),  
  file="models/CaliforniaGasLogNormalIncrPriorCGCana",  
  stanvars = stanvar(scode = myFuns, block = "functions"),  
  sample_prior = "only", seed = 123, iter = 200, chains = 2)
```



Review prior predictive output

California Cas Group: 200 prior predictive simulations



Run model with actual data

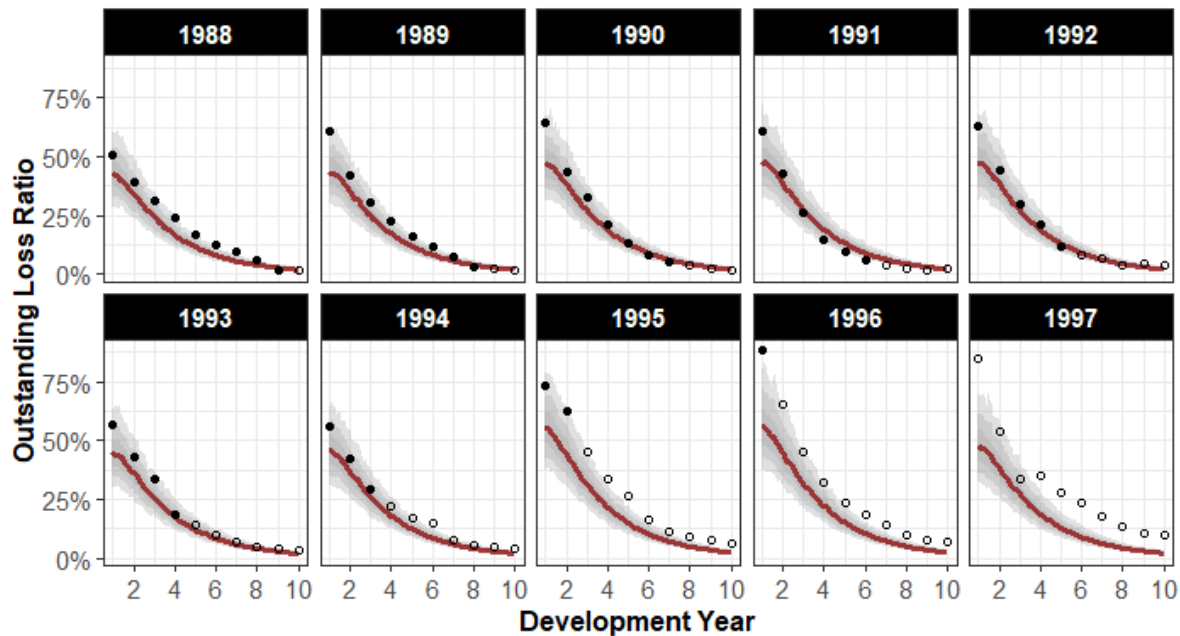
```
blafit <- update(bla, newdata=modDT_b[!is.na(loss_train)],  
                file="models/CaliforniaGasLogNormalIncrPosterior1CGCana",  
                sample_prior="no", seed=123, iter=500)
```



Outstanding data with holdouts

Outstanding Loss Ratio Development by Accident Year

50th, 80th and 95th Posterior Predictive Intervals

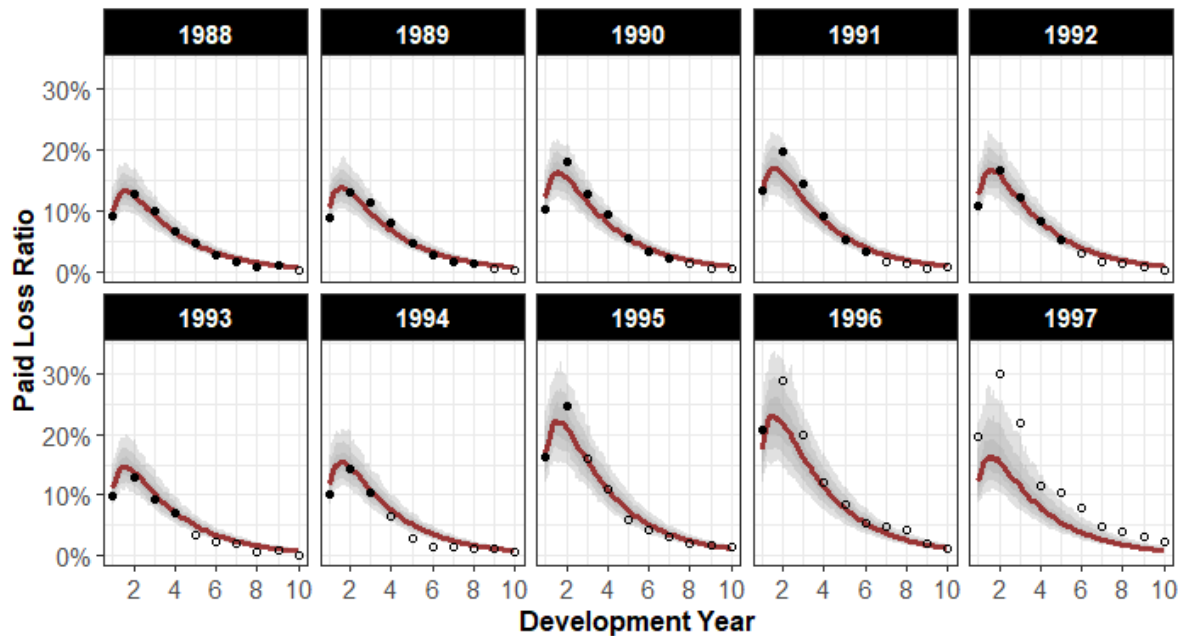


Institute
and Faculty
of Actuaries

Paid data with holdouts

Incremental Paid Loss Ratio Development by Accident Year

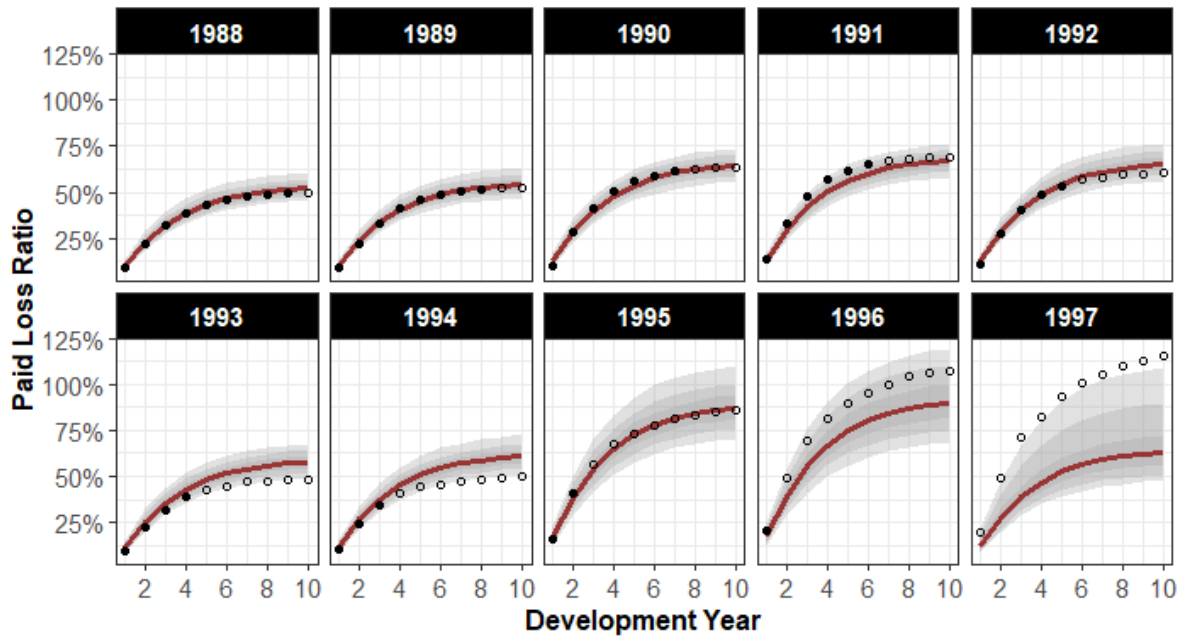
50th, 80th and 95th Posterior Predictive Intervals



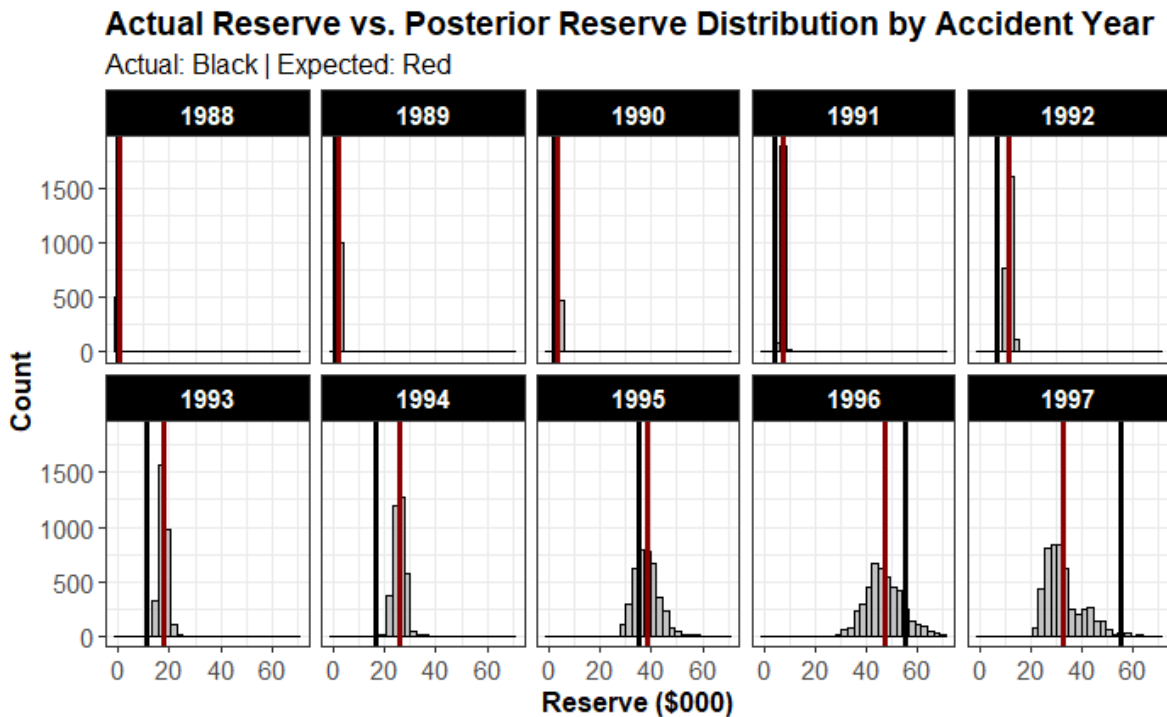
Cumulative Paid data with holdouts

Cumulative Paid Loss Ratio Development by Accident Year

50th, 80th and 95th Posterior Predictive Intervals



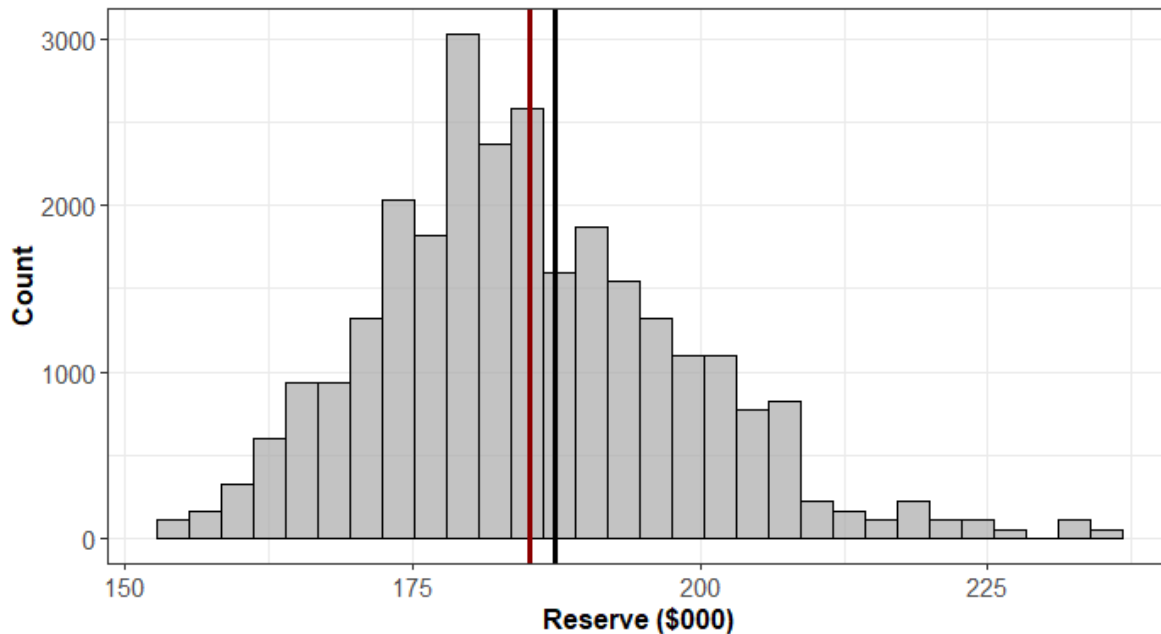
Distribution of future payments



Distribution of future payments

Actual Reserve vs. Posterior Reserve Distribution

Actual: Black | Expected: Red



From 500 samples:

Min.	151,776
1st Qu.	174,634
Median	183,273
Mean	184,184
3rd Qu.	191,748
Max.	236,694



Institute
and Faculty
of Actuaries



Institute
and Faculty
of Actuaries

Compartmental Reserving Models

Summary

25 April 2019

Summary

- Compartmental reserving models offer:
 - A flexible and transparent framework to develop parametric non-linear curves to describe development of outstanding and paid claims, simultaneously
 - Insight for a variety of small data sizes, as industry data and expert judgement can naturally be incorporated
 - Intuitive and transferable claims process-linked outputs, e.g. business plan LRs
- Bayesian modelling framework offers flexible approach to model process and parameter distribution
 - Expert judgement required to set prior assumptions and review model output
 - Paid development should be modelled on an incremental basis



References

- Bürkner, Paul-Christian. 2017. “brms: An R Package for Bayesian Multilevel Models Using Stan.” Journal of Statistical Software 80 (1): 1–28. doi:10.18637/jss.v080.i01. <https://paul-buerkner.github.io/brms/>
- Morris, Jake. 2016. “Hierarchical Compartmental Models for Loss Reserving.” In. Casualty Actuarial Society Summer E-Forum; <https://www.casact.org/pubs/forum/16sforum/Morris.pdf>



Questions

Comments

The views expressed in this [publication/presentation] are those of invited contributors and not necessarily those of the IFoA. The IFoA do not endorse any of the views stated, nor any claims or representations made in this [publication/presentation] and accept no responsibility or liability to any person for loss or damage suffered as a consequence of their placing reliance upon any view, claim or representation made in this [publication/presentation].

The information and expressions of opinion contained in this publication are not intended to be a comprehensive study, nor to provide actuarial advice or advice of any nature and should not be treated as a substitute for specific advice concerning individual situations. On no account may any part of this [publication/presentation] be reproduced without the written permission of the IFoA [*or authors, in the case of non-IFoA research*].



Institute
and Faculty
of Actuaries